Renormalizing Rectangles and Other Topics in Random Matrix Theory

Joshua Feinberg[1] and A. Zee[1]

We consider random Hermitian matrices made of complex or real $M \times N$ rectangular blocks, where the blocks are drawn from various ensembles. These matrices have N pairs of opposite real nonvanishing eigenvalues, as well as $M - N$ zero eigenvalues (for $M > N$). These zero eigenvalues are "kinematical" in the sense that they are independent of randomness. We study the eigenvalue distribution of these matrices to leading order in the large-N, M limit in which the "rectangularity" $r = M/N$ is held fixed. We apply a variety of methods in our study. We study Gaussian ensembles by a simple diagrammatic method, by the Dyson gas approach, and by a generalization of the Kazakov method. These methods make use of the invariance of such ensembles under the action of symmetry groups. The more complicated Wigner ensemble, which does not enjoy such symmetry properties, is studied by large-N renormalization techniques. In addition to the kinematical δ-function spike in the eigenvalue density which corresponds to zero eigenvalues, we find for both types of ensembles that if $|r - 1|$ is held fixed as $N \to \infty$, the N nonzero eigenvalues give rise to two separated lobes that are located symmetrically with respect to the origin. This separation arises because the nonzero eigenvalues are repelled macroscopically from the origin. Finally, we study the oscillatory behavior of the eigenvalue distribution near the endpoints of the lobes, a behavior governed by Airy functions. As $r \to 1$ the lobes come closer, and the Airy oscillatory behavior near the endpoints that are close to zero breaks down. We interpret this breakdown as a signal that $r \to 1$ drives a crossover to the oscillation governed by Bessel functions near the origin for matrices made of square blocks.

---

[1] Institute for Theoretical Physics, University of California, Santa Barbara, California 93106.

1. INTRODUCTION

In random matrix theory, a number of authors[1-8] have studied the eigenvalue distribution of a Hermitian matrix H of the form

$$H = \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix} \tag{1.1}$$

in which C is an $N \times N$ complex random matrix taken from an ensemble with the probability distribution

$$P(C) = \frac{1}{Z} \exp(-N \operatorname{Tr} C^\dagger C) \tag{1.2}$$

with N tending to infinity. These so-called chiral matrices appear in a variety of physical problems. For example, in quantum chromodynamics one typically integrates over the quarks and studies the so-called fermion determinant. The gluon fluctuations are then often treated approximately by saying that they effectively render the relevant matrix in the determinant random.[1,9,10] The chiral structure corresponds to left- and right-handed quarks. As another example, Hikami et al.[11-13] have proposed a model for electron scattering off impurities in quantum Hall fluids in the spin-degenerate limit. The blocks in (1.1) correspond to spin-up and spin-down electrons. In the same spirit, one may consider any problem involving random scattering between two groups of states, for example, between two cavities. As pointed out by Nagao and Slevin,[4] these matrices also appear in the study of transport in disordered conductors. In this paper, we study a slight generalization of this problem, with C taken to be an $M \times N$ rectangular matrix, with M and N both tending to infinity. For $M - N$ of order N^0, we expect the density of eigenvalues to be the same as for the $M = N$ case. Here we would like to study the case where the measure of rectangularity,

$$r \equiv M/N \tag{1.3}$$

is held fixed as both M and N tend to infinity. Some aspects of this problem have been studied before and we will note the appropriate references below. We denote the matrix elements of C by

$$C_{i\alpha}, \quad \text{where} \quad i = 1, 2, ..., M \quad \text{and} \quad \alpha = 1, 2, ..., N$$

With no loss of generality we assume throughout this paper that $M \geq N$, namely that $r \geq 1$. Our notation is such that Latin indices always run from

1 through M, whereas indices denoted by Greek letters run from 1 to N.[2] As a result of their specific structure these matrices have N pairs of opposite real nonvanishing eigenvalues, as well as $M - N$ zero eigenvalues. These zero eigenvalues are "kinematical" in the sense that they are independent of the probability distribution.

We derive the eigenvalue distribution of these matrices to leading order in the large-N, M approximation for various ensembles of random blocks. We consider random Hermitian matrices made of complex or real $M \times N$ rectangular blocks, where the blocks are drawn either from ensembles symmetric under some group action or from nonsymmetric ensembles. For concreteness, we specialize to Gaussian ensembles in the first case. In the second case we analyze matrices of the "Wigner class," namely blocks whose entries are drawn independently of one another from the probability distribution. We find, not surprisingly, that to leading order in the large-N, M approximation, all the ensembles we studied result in the same eigenvalue distribution. In addition to the kinematical δ-function spike in the eigenvalue density which corresponds to zero eigenvalues, we find that if $|r - 1|$ does not scale to zero as $N \to \infty$, the N nonzero eigenvalues give rise to two well-separated lobes that are located symmetrically with respect to the origin. For random Hermitian matrices that are not made of blocks, the qualitative universality of the Wigner semicircular eigenvalue distribution is well understood as a result of the competition between level repulsion and the fact that very large eigenvalues are suppressed. Similar arguments explain the universality of the eigenvalue distribution we observe here for matrices made of rectangular blocks. Each lobe arises qualitatively for the same reasons that lead to the semicircular distribution. In addition, separation of the two lobes arises because the nonzero eigenvalues are repelled from the origin by the macroscopic number $(M - N)$ of zero eigenvalues.

In this paper $C^\dagger C$ and $CC^\dagger$ are Hermitian nonnegative matrices of dimensions $N \times N$ and $M \times M$, respectively. We are interested in the expectation value of the resolvent

$$\hat{G}_{N,M}(z) = \frac{1}{N+M} \operatorname{Tr} \frac{1}{z-H} \tag{1.4}$$

A straightforward calculation then yields a simple relation between $\hat{G}_{N,M}(z)$ and the resolvents of $C^\dagger C$ and $CC^\dagger$,

$$\hat{G}_{N,M}(z) = \frac{z}{N+M} \left[\operatorname{Tr}_{(N)} \frac{1}{z^2 - C^\dagger C} + \operatorname{Tr}_{(M)} \frac{1}{z^2 - CC^\dagger} \right] \tag{1.5}$$

---

[2] We shall deviate slightly from this convention only in Section 2, where μ, ν will run over all possible $M + N$ values. No confusion should arise.

where the subscript on each trace indicates the dimension of the matrix which is being traced over. The z^2 dependence of the resolvents in (1.5) arises because the eigenvalues of H in (1.1) occur in real opposite pairs. Indeed, given an N-dimensional vector x and an M-dimensional vector y such that $\binom{x}{y}$ is an eigenvector of H for an eigenvalue λ, then $\binom{x}{-y}$ is an eigenvector for $-\lambda$. In other words the matrix H (the "Dirac" operator, with its "chiral" components C and $C^\dagger$) anticommutes with the "γ_5" matrix

$$\begin{pmatrix} 1_N & 0 \\ 0 & -1_M \end{pmatrix}$$

The cyclic property of the trace implies the basic relation

$$\mathrm{Tr}_{(M)} \frac{1}{z^2 - CC^\dagger} = \mathrm{Tr}_{(N)} \frac{1}{z^2 - C^\dagger C} + \frac{M - N}{z^2} \tag{1.6}$$

This relation reflects the fact that $C^\dagger C$ and $CC^\dagger$ share the same strictly positive eigenvalues, but the $M \times M$ matrix $CC^\dagger$ has additional $M - N$ zero eigenvalues.

Combining (1.5) and (1.6), we therefore arrive at the two alternative expressions

$$\hat{G}_{N,M}(z) = \left(\frac{M - N}{N + M} \right) \frac{1}{z} + \frac{2z}{M + N} \mathrm{Tr}_{(N)} \frac{1}{z^2 - C^\dagger C}$$

$$= \left(\frac{N - M}{N + M} \right) \frac{1}{z} + \frac{2z}{M + N} \mathrm{Tr}_{(M)} \frac{1}{z^2 - CC^\dagger} \tag{1.7}$$

which allow us to express $\hat{G}_{N,M}(z)$ solely in terms of either $C^\dagger C$ or in terms of $CC^\dagger$. For later use we introduce the notation

$$\hat{G}_N(w) = \frac{1}{N} \mathrm{Tr}_{(N)} \frac{1}{w - C^\dagger C}, \qquad \hat{G}_M(w) = \frac{1}{M} \mathrm{Tr}_{(M)} \frac{1}{w - CC^\dagger} \tag{1.8}$$

in terms of which we rewrite (1.7) as

$$\hat{G}_{N,M}(z) = \left(\frac{M - N}{N + M} \right) \frac{1}{z} + \left(\frac{2N}{M + N} \right) z \hat{G}_N(z^2)$$

$$= \left(\frac{N - M}{N + M} \right) \frac{1}{z} + \left(\frac{2M}{M + N} \right) z \hat{G}_M(z^2) \tag{1.9}$$

Throughout this paper $\hat{G}$ stands for an unaveraged resolvent. The corresponding averaged quantity will be denoted simply by G.

This paper is organized as follows. We will first apply a variety of methods to study the density of eigenvalues. In Section 2 we derive the density of states of matrices H whose rectangular blocks are drawn either from the unitary or from the orthogonal Gaussian ensemble, employing diagrammatic techniques. Section 3 is devoted to blocks with independent entries (which we referred to[14] as the "Wigner class"). This ensemble is more difficult to handle because of lack of symmetry. We overcome this difficulty by applying recursive manipulations of the large-N renormalization group.[15-17] We find that as far as the density of states is concerned, this ensemble falls (in the planar limit) into the same universality class as the symmetric ensembles. In the Appendix we provide a proof of the central limit theorem by means of the large-N renormalization group as yet another example of its usefulness. In Section 4 we present the Dyson gas approach to these issues. After completing our work we realized that the results we obtained following the Dyson gas approach already appeared in ref. 18. Nevertheless, we include this section here for the paper to be self-contained and also because Section 5 partly relies on it. In Section 5 we first generalize Kazakov's method[19] to rederive the results of Section 2, and then use this method to determine the oscillatory fine structure of the eigenvalue density in Section 2, close to its support endpoints. We find that this oscillatory behavior is governed by Airy functions. As $r \to 1$ the lobes come closer, and the Airy oscillatory behavior near the endpoints that are close to zero breaks down. We interpret this breakdown as a signal that in the limit $r \to 1$ drives a crossover to the oscillation near the origin in the density of eigenvalues of matrices made of square blocks, an oscillation governed by Bessel functions.

2. A DIAGRAMMATIC APPROACH

As a simple warm-up exercise, and in order to set the stage, let us first apply the by-now well-known diagrammatic method to derive the Green's function

$$G(z) = \frac{1}{N+M} \left\langle \mathrm{Tr}\, \frac{1}{z-H} \right\rangle \tag{2.1}$$

in the large-N, M limit. To this end, let us consider the averaged resolvent

$$G_\nu^\mu(z) = \left\langle \left(\frac{1}{z-H} \right)_\nu^\mu \right\rangle \tag{2.2}$$

where the indices μ and ν run over all possible $M + N$ values. The average in (2.2) is performed with respect to the Gaussian measure

$$P(C) = \frac{1}{Z} \exp[-\sqrt{NM}\, m^2 \operatorname{Tr} C^\dagger C] \qquad (2.3)$$

where

$$Z = \int \prod_{i=1}^{M} \prod_{\alpha=1}^{N} d\operatorname{Re} C_{i\alpha}\, d\operatorname{Im} C_{i\alpha} \exp[-\sqrt{NM}\, m^2 \operatorname{Tr} C^\dagger C] \qquad (2.4)$$

is the partition function. We have introduced a normalization factor of $\sqrt{NM}$ in (2.3) so as to be consistent with (1.2) in the $N = M$ case. This factor renders (2.3) and (2.4) manifestly symmetric under $M \leftrightarrow N$. Some other normalizations, not symmetrical under $M \leftrightarrow N$, can always be introduced by multiplying the parameter m^2 by an appropriate factor of $r = M/N$. Borrowing some terminology of gauge field theory, we may consider C, $C^\dagger$ as "gluons" (in zero space-time dimensions) and $G_\nu^\mu(z)$ as the propagator of "quarks" (with complex mass z) which couple to these "gluons." We now proceed to calculate $G_\nu^\mu(z)$ diagrammatically. The two-point correlator associated with (2.3) is clearly

$$\langle C_{i\alpha} C_{j\beta}^* \rangle = \frac{1}{m^2 \sqrt{MN}} \delta_{ij} \delta_{\alpha\beta} \qquad (2.5)$$

This expression is the gluon propagator. The bare quark propagator is simply $1/z$. The quark–quark–gluon vertex factor is 1. These Feynman rules are summarized in Fig. 1.
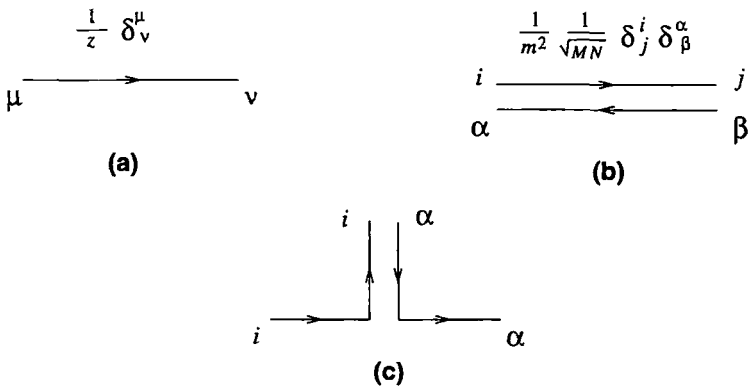

Fig. 1. The Feynman rules: (a) The bare quark propagator, (b) the gluon propagator, (c) the bare quark–quark–gluon vertex.
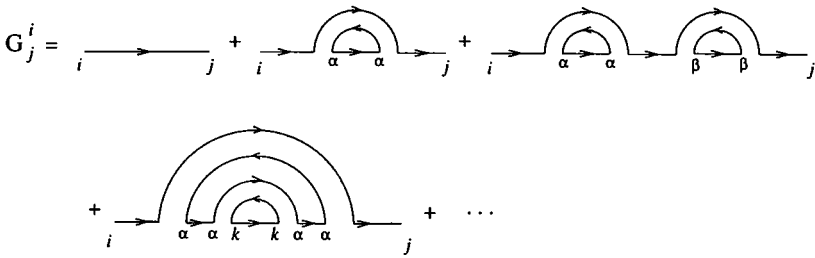
Fig. 2. The first few planar diagrams that contribute to G_j^i to leading order in the large-N, M approximation.

The weight (2.3) is Gaussian, so there are no gluonic self-interactions. Due to the block structure of H, the quark–quark–gluon vertex converts the color type carried by a quark. Namely, it converts a quark q_i into a quark q_α (and vice versa), but not into a quark q_j. Recall that the dominating diagrams in the large-N, M limit are all planar, and thus do not contain crossed color lines. Moreover, the (single) fermion line must always be at the boundary spanned by the planar graph. Consider now a typical planar Feynman diagram in the perturbative expansion of (2.2). Tracking the color indices through the diagram, we see that the rectangular off-diagonal blocks of (2.2) vanish identically

$$G_i^\alpha(z) = G_\alpha^i(z) = 0 \tag{2.6}$$

while the diagonal blocks are proportional to unit matrices,

$$G_j^i(z) = g_M(z)\,\delta_j^i, \qquad G_\beta^\alpha(z) = g_N(z)\,\delta_\beta^\alpha \tag{2.7}$$

Upon comparison with (1.8) we clearly have

$$g_N(z) = zG_N(z^2) \qquad \text{and} \qquad g_M(z) = zG_M(z^2) \tag{2.8}$$

Figure 2 shows some of the diagrams which contribute to $G_j^i(z)$ to leading order in large M, N.

A simple direct calculation shows that the off-diagonal blocks of $1/(z - H)$ are odd in C, $C^\dagger$ and thus do not contribute to (2.2), independently of the perturbation expansion. This conclusion clearly remains valid if we generalize (2.3) into any other probability distribution which is even in H, for example, a distribution of the form $P(C) = (1/Z) \exp[-\text{Tr } V(C^\dagger C)]$. The self-energy corrections $\Sigma_N(z)$ and $\Sigma_M(z)$ are defined as usual by

$$g_N(z) = \frac{1}{z - \Sigma_N(z)}, \qquad g_M(z) = \frac{1}{z - \Sigma_M(z)} \tag{2.9}$$
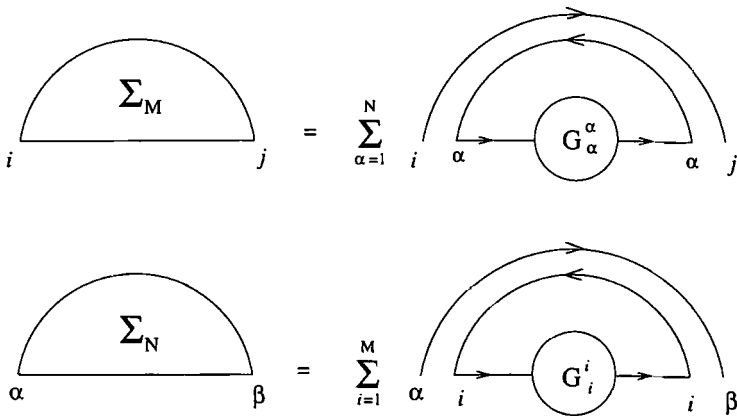
Fig. 3.   The two Schwinger–Dyson identities.

and correspond to the sum over all one-quark irreducible graphs contributing to (2.2), namely, the amputated quark propagator. For the Gaussian distribution (2.3), the propagators g_N, g_M and self-energies Σ_N, Σ_M are related by the simple Schwinger–Dyson identities, which we display diagrammatically in Fig. 3.

We thus have from Figs. 1 and 3

$$\Sigma_N(z) = \frac{1}{m^2 \sqrt{NM}} \sum_{i=1}^{M} G_i^i(z) = \frac{\sqrt{r}}{m^2} g_M(z) \tag{2.10}$$

and similarly,

$$\Sigma_M(z) = \frac{1}{m^2 \sqrt{r}} g_N(z) \tag{2.11}$$

We substitute the last two equations into (2.9) and obtain the two coupled equations

$$\begin{cases} g_N(z) = \left[z - \dfrac{\sqrt{r}}{m^2} g_M(z) \right]^{-1} \\[2mm] g_M(z) = \left[z - \dfrac{1}{m^2 \sqrt{r}} g_N(z) \right]^{-1} \end{cases} \tag{2.12}$$

for g_N and g_M. These two equations clearly transform into one another under $r \to 1/r$, which interchanges g_N and g_M [this is as it should be if the normalization factor in (2.3) is symmetrical under $M \leftrightarrow N$, that is, if m^2 is independent of r]. By definition, both propagators behave as $1/z$ in the

asymptotic region $z \to \infty$. This asymptotic behavior picks up the physical solution of the quadratic equations for g_N and g_M, and we find

$$g_N(z) = \frac{2}{(a-b)^2} \frac{1}{z} [z^2 - ab - \sqrt{(z^2 - a^2)(z^2 - b^2)}]$$

$$g_M(z) = \frac{2}{(a+b)^2} \frac{1}{z} [z^2 + ab - \sqrt{(z^2 - a^2)(z^2 - b^2)}]$$

(2.13)

where

$$a = \frac{1}{m}(r^{1/4} + r^{-1/4}), \qquad b = \frac{1}{m}(r^{1/4} - r^{-1/4}) \tag{2.14}$$

Note that b measures of the deviation of rectangles from squares: for $r = 1$, b vanishes. The Green's function (2.1) is thus given by

$$G(z) = \frac{1}{N+M} \sum_{\mu=1}^{N+M} G_\mu^\mu(z) = \frac{g_N(z) + r g_M(z)}{1+r} \tag{2.15}$$

where we used (2.2) and (2.7). Finally, upon substituting (2.13) into (2.15), we find that the averaged Green's function of H is

$$G(z) = \frac{2}{a^2 + b^2} \frac{1}{z} [z^2 - \sqrt{(z^2 - b^2)(z^2 - a^2)}] \tag{2.16}$$

Let us inspect now some of the features of (2.16). As we discussed in the introduction, H has $M - N$ "kinematical" zero eigenvalues, regardless of any ensemble averaging. In contrast, $C^\dagger C$ on the average does not have any zero eigenvalues, as we have already discussed. Thus, by definition, $G_{N,M}(z)$ has a simple pole at $z = 0$ with residue $(M-N)/(M+N) = (r-1)/(r+1)$, which is the first term on the right side of (1.9). As we can see from (2.14), our expression (2.16) clearly satisfies this condition provided $[(z^2 - b^2)(z^2 - a^2)]^{1/2} \to -ab$ as $z \to 0$. This sign of the square root corresponds to drawing all four cuts associated with the square root to the left of the branch point out of which they emanate. In addition, (2.14) is consistent with the required asymptotic behavior $1/z$ of (2.16) as $z \to \infty$. The averaged eigenvalue density of H is the discontinuity in (2.16) as we cross the real axis, except for the origin, which contains the "kinematical" zero eigenvalues of H. It is therefore given by

$$\rho(\lambda) = \frac{r-1}{r+1} \delta(\lambda) + \frac{2}{\pi |\lambda|} \frac{\theta[(a^2 - \lambda^2)(\lambda^2 - b^2)]}{a^2 + b^2} \sqrt{(a^2 - \lambda^2)(\lambda^2 - b^2)} \tag{2.17}$$

The Green's function (2.16) corresponds to the Hamiltonian (1.1). With only little more effort it is possible to generalize our discussion to calculate the Green's function of the Hamiltonian

$$H = \begin{pmatrix} \varepsilon & C^\dagger \\ C & -\varepsilon \end{pmatrix} \tag{2.18}$$

where ε is a fixed "energy." The off-diagonal blocks fluctuate as before. This Hamiltonian may describe, for example, tunneling between two energy levels with degeneracies N and M that are separated by an energy difference 2ε. For such a Hamiltonian (1.5) is modified into

$$G_{N,M}(z) = \frac{z+\varepsilon}{r+1} G_N(w) + \frac{r(z-\varepsilon)}{r+1} G_M(w) \tag{2.19}$$

where

$$w = z^2 - \varepsilon^2$$

such that the identifications (2.8) become

$$g_N(z) = (z+\varepsilon) G_N(w) \qquad \text{and} \qquad g_M(z) = (z-\varepsilon) G_M(w) \tag{2.20}$$

The bare quark propagator in Fig. 1 is split into two pieces, namely $1/(z-\varepsilon)$ for quarks carrying a $U(N)$ color index and $1/(z+\varepsilon)$ for quarks carrying a $U(M)$ index. The definitions in (2.9) change accordingly into

$$g_N(z) = \frac{1}{z-\varepsilon-\Sigma_N(z)}, \qquad g_M(z) = \frac{1}{z+\varepsilon-\Sigma_M(z)} \tag{2.21}$$

The Schwinger–Dyson identities (2.10) and (2.11) are unchanged. We note that the set of equations (2.10), (2.11), and (2.21) are now invariant under $r \to 1/r$ and $\varepsilon \to -\varepsilon$, which permutes the two energy levels in H and thus interchanges g_N and g_M. With these observation it is straightforward to see that (2.13) becomes

$$g_N(z) = \frac{2}{(a-b)^2} \frac{1}{z-\varepsilon} [w - ab - \sqrt{(w-a^2)(w-b^2)}]$$

$$\tag{2.22}$$

$$g_M(z) = \frac{2}{(a+b)^2} \frac{1}{z+\varepsilon} [w + ab - \sqrt{(w-a^2)(w-b^2)}]$$

with the same a and b as before. Thus, (2.15) and (2.16) finally become

$$G(z) = \frac{2}{a^2 + b^2} \frac{1}{w} \{ z[w - \sqrt{(w - a^2)(w - b^2)}] - ab\varepsilon \} \qquad (2.23)$$

which is manifestly invariant under a permutation of the two energy levels of H. Note that the matrix (2.18) has precisely $M - N$ "kinematical" (i.e., independent of the ensemble for C) eigenvectors which correspond to eigenvalue $-\varepsilon$. This means that (2.23) has a simple pole at $z = -\varepsilon$ with residue $(r-1)/(r+1)$ and no pole at $z = +\varepsilon$. This property holds provided $[(w - a^2)(w - b^2)]^{1/2} \to -ab$ as $w \to 0$, which we already encountered in our analysis of the $\varepsilon = 0$ case. The eigenvalue distribution corresponding to (2.23) is therefore

$$\rho(\lambda) = \frac{r-1}{r+1} \delta(\lambda + \varepsilon) + \frac{\theta[(a^2 + \varepsilon^2 - \lambda^2)(\lambda^2 - b^2 - \varepsilon^2)]}{\pi(a^2 + b^2)}$$

$$\times \frac{2|\lambda|}{\lambda^2 - \varepsilon^2} \sqrt{(a^2 + \varepsilon^2 - \lambda^2)(\lambda^2 - b^2 - \varepsilon^2)} \qquad (2.24)$$

In the limit $(m\varepsilon) \to \infty$, the randomness in (2.18) is suppressed, and (2.24) should reproduce the eigenvalue distribution of the deterministic part of (2.18). This is indeed the case. In the limit we have a/ε, $b/\varepsilon \to 0$, so both lobes in (2.24) shrink. Each lobe contains N eigenvalues, whose number is preserved as the lobes shrink to zero width, and thus produce δ-function spikes of strength $N/(M + N)$ each. The right lobe produces in this way a spike at $\lambda = \varepsilon$, while the left lobe coalesces with the already existing $\delta(\lambda + \varepsilon)$ spike in (2.24), which contains $M - N$ eigenvalues, and thus produces a spike containing M eigenvalues.

3. BLOCKS WITH INDEPENDENT MATRIX ELEMENTS AND THEIR RENORMALIZATION GROUP ANALYSIS

It is rather difficult to apply the diagrammatic method and sum all the planar diagrams that contribute to $G(z)$ in case of non-Gaussian ensembles. For such ensembles that are invariant under the action of some symmetry group one may invoke other methods. However, these methods are inapplicable to ensembles lacking the action of a symmetry group.

A class of block structure random matrix models that is not unitary (or orthogonal) invariant involves matrix blocks C in which each matrix element $C_{i\alpha}$ is randomly distributed independently of the others, with the same distribution for all matrix elements. We normalize the matrix

elements $C_{i\alpha}$ symmetrically with respect to M and N, such that the two-point correlator

$$\langle C_{i\alpha} C_{j\beta}^* \rangle = \frac{\sigma^2}{\sqrt{MN}} \delta_{ij} \delta_{\alpha\beta} \tag{3.1}$$

of this probability distribution would coincide with the two-point correlator of the Gaussian distribution

$$d\mu(C) \sim \exp\left[-\frac{\sqrt{MN}}{\sigma^2} \operatorname{Tr} C^\dagger C \right]$$

For notational simplicity we replaced here the m^{-2} in (2.3) by σ^2. For this class of matrix models the method of orthogonal polynomials is not directly applicable, and alternative methods should be sought for.

For concreteness as well as for simplicity, we consider below the probability distribution in which $C_{i\alpha}$ may take one of the two values

$$\pm \frac{\sigma}{(NM)^{1/4}} \tag{3.2}$$

with equal probability, where σ is a finite number. However, it will be clear from the discussion below that our conclusions are not limited to this particular distribution. In order to keep our formulas generic, we therefore treat the $C_{i\alpha}$ as complex numbers, as long as we do not utilize (3.2) explicitly.

For this ensemble $|C_{i\alpha}|^2 = \sigma^2/\sqrt{MN}$ deterministically, and thus in particular the diagonal matrix elements of $C^\dagger C$ and $CC^\dagger$ do not fluctuate and are given by

$$(C^\dagger C)_{\alpha\alpha} = \sigma^2 \sqrt{r} \qquad \text{and} \qquad (CC^\dagger)_{ii} = \sigma^2/\sqrt{r} \tag{3.3}$$

We use this convenient property of (3.2) in our calculations below. This, however, is done with no loss of generality, because in the generic case the nonfluctuating quantities in (3.3) should simply be replaced by their averages, which are given on the right-hand side of (3.3).

The random matrix distribution we consider is a generalization of the very first model studied by Wigner[20] to random matrices with block structure. Indeed, Wigner originally considered large $N \times N$ random Hermitean matrices ϕ, whose elements $\phi_{ij} = \phi_{ji}^*$ were either $+\sigma/\sqrt{N}$ or $-\sigma/\sqrt{N}$, with equal probability. This matrix model follows a semicircle law for the density of eigenvalues. This semicircular profile of the eigenvalue density was rederived recently[15] using a large-N "renormalization-group"-inspired

approach.[16] In what follows, we apply the same method to find the eigen-value density $\rho(\lambda)$ of the random block matrices with independent entries introduced above.

We are interested in

$$G(z) = \lim_{N, M \to \infty} G_{N, M}(z) \tag{3.4}$$

from which $\rho(\lambda) = (1/\pi) \operatorname{Im} G(\lambda - i\varepsilon)$ may be extracted immediately. We start our "renormalization group" calculation by trying to relate $G_{N+1, M}(z)$ to $G_{N, M}(z)$. To this end we consider the $M \times (N+1)$ block

$$C' = (C, v)$$

where v is an M-dimensional vector. By definition, each element of C' may now take one of the two values $\pm \sigma/[(N+1) M]^{1/4}$ with equal probability. A comparison with the original $N \times M$ block suggests then that we may draw the $C'_{i\alpha}$ from (3.2) provided we rescale the σ parameter in that equation into

$$\sigma' = \sigma \left(\frac{N}{N+1} \right)^{1/4} \tag{3.5}$$

The nonfluctuating norm squared of v is then given by

$$v^\dagger v \equiv (C'^\dagger C')_{N+1, N+1} = \sigma'^2 \sqrt{r} \tag{3.6}$$

Following ref. 15, we obtain[3] after some straightforward algebra

$$(N+1) \, \hat{G}_{N+1}(w) \equiv \operatorname{Tr}_{(N+1)} \frac{1}{w - C'^\dagger C'}$$

$$= \operatorname{Tr}_{(N)} \frac{1}{w - C^\dagger C - C^\dagger [v \otimes v^\dagger/(w - v^\dagger v)] \, C}$$

$$+ \frac{1}{w - v^\dagger v - v^\dagger C [1/(w - C^\dagger C)] \, C^\dagger v} \tag{3.7}$$

where $w = z^2$. We now average over the distribution governing C', keeping terms up to $\mathcal{O}(N^0)$. We first average over the components of v.

Expanding the two fractions on the right-hand side of (3.7) into geometric series, we see that we have to average products of the form

---

[3] The analogue of $v^\dagger v$ in ref. 15 was a quantity of $\mathcal{O}(1/N)$ which was therefore neglected in the $N \to \infty$ limit. Here $v^\dagger v$ is a finite number and therefore must be retained.

$v_i^* v_j v_k^* v_l \cdots v_p^* v_q$, which contract against products of elements of matrices independent of the v_i. Clearly,

$$\langle v_i^* v_j \rangle = \langle C_{i,N+1}'^* C_{j,N+1}' \rangle = \frac{\sigma'^2}{\sqrt{MN}} \delta_{ij}$$

simply produces a single trace, multiplied by $\sigma'^2/\sqrt{MN}$. The next non-vanishing correlator is

$$\langle v_i^* v_j v_k^* v_l \rangle = \frac{\sigma'^4}{NM} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \delta_{ijkl}) \tag{3.8}$$

where $\delta_{ijkl} = 1$ when $i = j = k = l$ and 0 otherwise. The last piece in (3.8) is by definition the fourth-order cumulant of the distribution, added to the usual pairs of Wick contractions. The correlator (3.8) then contracts against two matrices in the geometrical series mentioned above, producing a term of order $(\sigma'^2/\sqrt{NM})^2$. In the large-N, M limit, the dominating term in this contraction is the term with the maximal independent index summations, which amounts here to two traces. These two traces are produced here only by a single pair of Wick contractions. The other pair of Wick contractions (which produces only a single trace) as well as the fourth-order cumulant are therefore negligible in the large-N, M limit and may be discarded to leading order. This structure persists for correlators of higher order. The $2n$-point correlator produces in the geometric series a term proportional to $(\sigma'^2/\sqrt{NM})^n$. In that term, a unique string of n Wick contractions produces the maximal number n of traces, and therefore dominates the large-N, M limit. At this point it becomes clear why our calculation and therefore the results it leads to are insensitive to the details of the distribution of the $C_{i\alpha}$. Clearly, only Wick contractions dominate these averages in this limit, and thus only the two-point function (3.1) of the distribution matters. This insensitivity to the details of the distribution was checked explicitly in ref. 15, where various distributions led to the same result. The leading terms in the geometric series may be resummed, and one finds the v average

$$\left\langle \mathrm{Tr}_{(N+1)} \frac{1}{w - C'^\dagger C'} \right\rangle_r$$

$$= \mathrm{Tr}_{(N)} \frac{1}{w - C^\dagger C} + \frac{\partial}{\partial w} \log \left[w + (N - M) \frac{\sigma'^2}{\sqrt{NM}} \right.$$

$$\left. - \frac{\sigma'^2}{\sqrt{NM}} w \, \mathrm{Tr}_{(N)} \frac{1}{w - C^\dagger C} \right] \tag{3.9}$$

Invoking large-N factorization, we can average over the remaining block C immediately, by replacing $\hat{G}_N$ inside the logarithm in (3.9) by its average. Thus,

$$(N+1)\, G_{N+1}(w)$$

$$= NG_N(w) + \frac{\partial}{\partial w} \log\left[w + \frac{(1-r)\,\sigma'^2}{\sqrt{r}} - \frac{\sigma'^2}{\sqrt{r}}\, wG_N(w) \right] \qquad (3.10)$$

Remarkably, in the large-N, M limit, the v average of $\hat{G}_{N+1}(w)$ involves only $\hat{G}_N(w)$, and thus (3.10) is indeed a local (along the N axis) recursion relation involving only G_N-type Green's functions. This means that the large-N "renormalization group" recursions for the full Green's function $G_{N,M}$ close among themselves, as we now show.

Combining (1.9) and (3.10), we obtain the recursion relation for the complete averaged Green's function (1.4)

$$(N+M+1)\, G_{N+1,M}(z, \sigma') - (N+M)\, G_{N,M}(z, \sigma')$$

$$= \frac{\partial}{\partial z} \log\left[z + \frac{(1-r)\,\sigma'^2}{2z\sqrt{r}} - \frac{(r+1)\,\sigma'^2}{2\sqrt{r}}\, G_{N,M}(z, \sigma') \right] \qquad (3.11)$$

where we have displayed the explicit σ' parameter associated with the larger C' block.

In the large-N limit (3.5) becomes $\sigma' = \sigma - \sigma/(4N) + \cdots$. In this limit, the only possible explicit N, M dependence in $G_{N,M}$ is through the finite ratio $r = M/N$. Therefore we may write the left-hand side of (3.11) as

$$[(N+M+1)\, G_{N+1,M}(z, \sigma') - (N+M)\, G_{N,M}(z, \sigma)]$$

$$+ (N+M)[G_{N,M}(z, \sigma) - G_{N,M}(z, \sigma')]$$

$$= \frac{\partial}{\partial N}[(N+M)\, G_{N,M}(z, \sigma)] + \frac{N+M}{4N}\, \sigma\, \frac{\partial}{\partial \sigma}\, G_{N,M}(z, \sigma)$$

$$= G_{N,M}(z, \sigma) - r(r+1)\frac{\partial}{\partial r}\, G_{N,M}(z, \sigma) + \frac{r+1}{4}\, \sigma\, \frac{\partial}{\partial \sigma}\, G_{N,M}(z, \sigma) \qquad (3.12)$$

To leading order in $1/N$ we may drop the N, M indices of the Green's function, replacing it by its asymptotic limit (3.4), and replace σ' by σ inside the

logarithm in (3.11). The recursion relation (3.11) thus becomes a partial differential equation

$$G(z, \sigma) - r(r+1) \frac{\partial}{\partial r} G(z, \sigma) + \frac{r+1}{4} \sigma \frac{\partial}{\partial \sigma} G(z, \sigma)$$

$$= \frac{\partial}{\partial z} \log \left[z + \frac{(1-r) \sigma^2}{2z \sqrt{r}} - \frac{(r+1) \sigma^2}{2 \sqrt{r}} G(z, \sigma) \right] \qquad (3.13)$$

It is easy to see from (1.4) and (3.4) that $G(z, \sigma)$ satisfies the simple scaling rule

$$G(z, \sigma) = \frac{1}{\sigma} G \left(\frac{z}{\sigma}, 1 \right) \qquad (3.14)$$

which implies that

$$\sigma \frac{\partial}{\partial \sigma} G(z, \sigma) = - z \frac{\partial}{\partial z} G(z, \sigma) - G(z, \sigma) \qquad (3.15)$$

Thus, using (3.15) to eliminate $\sigma(\partial/\partial \sigma) G(z, \sigma)$ from (3.13), we arrive at the final form of our differential equation for $G(z, \sigma)$, namely,

$$\frac{3-r}{4} G(z, \sigma) - r(r+1) \frac{\partial}{\partial r} G(z, \sigma) - \frac{r+1}{4} z \frac{\partial}{\partial z} G(z, \sigma)$$

$$= \frac{\partial}{\partial z} \log \left[z + \frac{(1-r) \sigma^2}{2z \sqrt{r}} - \frac{(r+1) \sigma^2}{2 \sqrt{r}} G(z, \sigma) \right] \qquad (3.16)$$

This equation tells us how a change in z can be compensated by a change in the rectangularity r.

As a consistency check of our results we can repeat the recursive procedure discussed above, but instead of adding an M-dimensional column vector to C, we add to it an N-dimensional row vector u, creating an $(M + 1) \times N$ block C''

$$C'' = \begin{pmatrix} C \\ u \end{pmatrix}$$

The recursion relation in this case connects, in the large-N, M limit, $G_{M+1}(w)$ and $G_M(w)$, and therefore relates $G_{N, M+1}(z)$ to $G_{N, M}(z)$. Thus,

we simply interchange $N \leftrightarrow M$ in all steps of our calculation above, namely, $r \leftrightarrow 1/r$. The differential equation for $G(z, \sigma)$ we derived from this recursion reads

$$\frac{3r - 1}{4r} G(z, \sigma) + (r + 1) \frac{\partial}{\partial r} G(z, \sigma) - \frac{r + 1}{4r} z \frac{\partial}{\partial z} G(z, \sigma)$$

$$= \frac{\partial}{\partial z} \log\left[z - \frac{(1 - r) \sigma^2}{2z \sqrt{r}} - \frac{(r + 1) \sigma^2}{2 \sqrt{r}} G(z, \sigma) \right] \qquad (3.17)$$

which is indeed the transform of (3.16) under $r \to 1/r$.

The fact that $G(z, \sigma)$ satisfies both (3.16) and its transform under $r \to 1/r$ means that $G(z, \sigma, r) = G(z, \sigma, 1/r)$. This r-inversion symmetry of G should be anticipated from our N, M-symmetric definition of the probability distribution (3.1) in the first place. An important consequence of this r-inversion symmetry is that $(\partial/\partial r) G$ vanishes[4] at $r = 1$. Thus, at the point $r = 1$, i.e., for Hamiltonians made of square blocks, (3.16) reduces to the differential equation

$$G(z, \sigma) - z \frac{\partial}{\partial z} G(z, \sigma) = 2 \frac{\partial}{\partial z} \log[z - \sigma^2 G(z, \sigma)] \qquad (3.18)$$

previously derived in ref. 15, as expected.

As was stated at the beginning of this section, only the two-point correlator (3.1) of the random matrix distribution was relevant in the derivation of (3.16). Hence, the Green's function $G(z)$ of any distribution obeying (3.1) is a solution of (3.16). We have thus shown that for the Wigner ensemble, $G(z)$ and the density of eigenvalues are universal. In particular, the complex Hermitian distribution (2.3) as well as the real symmetric distribution (4.16) of the previous section respect (3.1) upon the identification $m^2 = \sigma^{-2}$. Thus, their Green's function (2.16) must be a solution of (3.16). A simple check verifies that this is indeed the case. Therefore, the density of eigenvalues $\rho(\lambda) = (1/\pi) \operatorname{Im} G(\lambda - i\varepsilon)$ is given by (2.17).

As yet another example of the usefulness of the large-N renormalization group we use it to prove the central limit theorem in the Appendix.

By a simple power counting argument (see Section 2 of ref. 14, and also ref. 17), it is straightforward to extend the diagrammatic method of the previous section to treat the probability distribution considered in this section as well.

---

[4] This is simply because if $f(r) = f(r^{-1})$, then $(\partial/\partial r) f(r) = -r^{-2}(\partial/\partial r^{-1}) f(r^{-1})$, and therefore $f'(1) = -f'(1) = 0$.

4. DYSON GAS APPROACH

In this section we present the Dyson gas approach to study the eigenvalue distribution of matrices made of rectangular blocks. After completing our calculations we realized that our results were previously obtained by Periwal et al.[18] We assume that the $M \times N$ rectangular blocks $C_{i\alpha}$ of the Hamiltonian H in (1.1) admit the action of some symmetry group. Here we focus on blocks with complex entries, but we will state some results concerning blocks with real entries in the end. The complex blocks are endowed with the natural $U(M) \times U(N)$ action

$$C \to VCU, \qquad V \in U(M), \quad U \in U(N) \tag{4.1}$$

One can use this action to bring C to the form

$$C = \begin{pmatrix} \Lambda_N \\ 0_{(M-N) \times N} \end{pmatrix} \tag{4.2}$$

where Λ_N is a real diagonal $N \times N$ matrix $\mathrm{diag}(\lambda_1,..., \lambda_N)$. Therefore, the Hermitian matrix H in (1.1) is a generator of the symmetric space $U(M+N)/U(M) \otimes U(N)$. From these considerations it is clear that $C^\dagger C$ may be diagonalized into $\mathrm{diag}(\lambda_1^2,..., \lambda_N^2)$ and $CC^\dagger$ into the same form, but with additional $M-N$ zeros, in accordance with (1.6). The probability distribution has to be invariant under (4.1). Here we consider distributions of the form

$$P(C) = \frac{1}{Z} \exp[-\sqrt{MN} \, \mathrm{Tr} \, V(C^\dagger C)] \tag{4.3}$$

where V is a polynomial and Z is the partition function of these matrices.

We are interested only in averages of quantities that are invariant under (4.1). We thus transform from the Cartesian coordinates $C_{i\alpha}$ to polar coordinates V_{ij}, $U_{\alpha\beta}$, and λ_α. Integrations over the unitary groups are irrelevant in calculating averages of invariant quantities, which involve only the eigenvalues $s_\alpha = \lambda_\alpha^2$ of $C^\dagger C$.

The partition function for these eigenvalues then reads[18]

$$Z = \prod_{\alpha=1}^{N} \int_0^\infty ds_\alpha \exp[-\sqrt{NM} \, V(s_\alpha)] \prod_{\beta=1}^{N} s_\beta^{M-N} \prod_{1 \leqslant \gamma < \delta \leqslant N} (s_\gamma - s_\delta)^2 \tag{4.4}$$

The last two products constitute the Jacobian associated with polar coordinates. In particular, $\prod (s_\gamma - s_\delta)^2$ is the familiar Vandermonde determinant.

The other product is a feature peculiar to nonsquare blocks. As a trivial check of the validity of (4.4), note that the integration measures in (2.4) and (4.4) have the same scaling dimension under $C \to \xi C$, $\xi > 0$.

Following Dyson, we observe that (4.4) may be interpreted as the partition function for a one-dimensional gas of particles whose coordinates are given by the eigenvalues s_α. The integrand in (4.4) may be expressed as $\exp[-\sqrt{NM}\,\mathscr{E}]$, where

$$\mathscr{E} = \sum_{\alpha=1}^{N} \left[V(s_\alpha) - \frac{r-1}{\sqrt{r}} \log s_\alpha \right] - \frac{1}{N\sqrt{r}} \sum_{1 \leqslant \alpha < \beta \leqslant N} \log(s_\alpha - s_\beta)^2 \qquad (4.5)$$

is the energy functional of the Dyson gas. In the large-N, M limit (4.4) is governed by the saddle point of (4.5), namely by a $C^\dagger C$ eigenvalue distribution $\{s_\alpha\}$ that satisfies

$$\frac{\partial \mathscr{E}}{\partial s_\alpha} = V'(s_\alpha) - \frac{r-1}{\sqrt{r}} \frac{1}{s_\alpha} - \frac{2}{N\sqrt{r}} \sum_{\beta=1}^{N}{}' \frac{1}{s_\alpha - s_\beta} \qquad (4.6)$$

Here the prime over the sum symbol indicates that $\beta = \alpha$ is excluded from the sum.

We now turn our attention to the average eigenvalue density of H, which we may readily deduce[21] from the averaged Green's function $G_{N,M}(z)$ in (1.4). The s_α are eigenvalues of $C^\dagger C$. We thus calculate first $G_N(z^2)$, which, according to (1.8), is given by

$$G_N(w) = \frac{1}{N} \sum_{\alpha=1}^{N} \left\langle \frac{1}{w - s_\alpha} \right\rangle \qquad (4.7)$$

Here the angular brackets denote averaging with respect to (4.4). By definition, $G_N(w)$ behaves asymptotically as

$$G_N(w) \xrightarrow[w \to \infty]{} \frac{1}{w} \qquad (4.8)$$

It is clear from (4.7) that for $s > 0$, $\varepsilon \to 0+$ we have

$$G_N(s - i\varepsilon) = \frac{1}{N} \text{P.P.} \sum_{\alpha=1}^{N} \left\langle \frac{1}{s - s_\alpha} \right\rangle + \frac{i\pi}{N} \sum_{\alpha=1}^{N} \langle \delta(s - s_\alpha) \rangle \qquad (4.9)$$

where P.P. stands for the principal value. Therefore the average eigenvalue density of $C^\dagger C$ is given by $(1/\pi)\,\mathrm{Im}\,G_N(s-i\varepsilon)$. In the large-$N$, M limit, the real part of (4.9) is fixed by (4.6), namely

$$\mathrm{Re}\,G_N(s-i\varepsilon)=\frac{1}{2}\left[\sqrt{r}\,V'(s)-(r-1)\frac{1}{s}\right] \tag{4.10}$$

The potential $V(s)$ in (4.3) clearly has at least one minimum for $s>0$, and will therefore cause the eigenvalues to coalesce into a single finite band or more along the real positive axis. Moreover, the $\log s$ term in (4.5) clearly implies that the $\{s_\alpha\}$ are repelled from the origin. We thus anticipate that the lowest band will be located at a finite distance from the origin $s=0$.

At this point we depart from discussing the general distribution and assume for simplicity that the probability distribution is given by the Gaussian distribution (2.3) with

$$V(s)=m^2 s \tag{4.11}$$

In this case we expect the $\{s_\alpha\}$ to be contained in the single finite segment $0<b^2<s<a^2$, with $a>b>0$ yet to be determined.[5] This means that $G_N(w)$ should have a cut connecting b^2 and a^2. This conclusion, together with (4.10), implies that $G_N(w)$ must be of the form

$$G_N(w)=\frac{1}{2}\left[\sqrt{r}\,m^2-(r-1)\frac{1}{w}\right]+F(w)\sqrt{(w-b^2)(w-a^2)}$$

where $F(w)$ is analytic in the w plane (with the origin excluded.) The asymptotic behavior (4.8) then fixes

$$F(w)=-\frac{\sqrt{r}\,m^2}{2w},\qquad a^2+b^2=\frac{2}{m^2}\left(\sqrt{r}+\frac{1}{\sqrt{r}}\right) \tag{4.12}$$

and thus

$$G_N(w)=\frac{1}{2w}\left[\sqrt{r}\,m^2 w-r+1-\sqrt{r}\,m^2\sqrt{(w-b^2)(w-a^2)}\right] \tag{4.13}$$

The eigenvalue distribution of $C^\dagger C$ is therefore

$$\tilde{\rho}(s)=\frac{1}{\pi}\,\mathrm{Im}\,G_N(s-i\varepsilon)=\frac{\sqrt{r}\,m^2}{2\pi s}\sqrt{(s-b^2)(a^2-s)} \tag{4.14}$$

for $b^2<s<a^2$, and zero elsewhere.

[5] We find below, of course, that a and b coincide with the expressions in (2.14).

We now substitute $G_N(z^2)$ from (4.13) into (1.9) to obtain an expression for $G_{N,M}(z)$. As we discussed in the introduction and in Section 2, $G_{N,M}(z)$ has a simple pole at $z = 0$ with residue $(M-N)/(M+N) = (r-1)/(r+1)$, which is the first term on the right side of (1.9). We thus conclude from (1.9) that $wG_N(w)$ must vanish at $w = 0$, which in turn implies a second condition[6] on a, b, namely

$$ab = \frac{r-1}{m^2\sqrt{r}} \tag{4.15}$$

We are now able to fix a and b from (4.12) and (4.15) and find that they are given by (2.14). We thus find that $G_{N,M}(z)$ coincides with (2.16) and that the averaged eigenvalue density of H is the expression in (2.17).

We close this section by sketching a similar analysis of Gaussian random Hamiltonians made of real $M \times N$ blocks C. We parametrize the Gaussian real orthogonal ensemble by

$$P(C) = \frac{1}{Z} \exp\left[-\frac{m^2}{2} \sqrt{NM}\, \mathrm{Tr}\, C^T C \right] \tag{4.16}$$

with the partition function

$$Z = \int \prod_{i=1}^{M} \prod_{\alpha=1}^{N} dC_{i\alpha} \exp\left[-\frac{m^2}{2} \sqrt{NM}\, \mathrm{Tr}\, C^T C \right] \tag{4.17}$$

The two-point correlator associated with (4.16) is clearly

$$\langle C_{i\alpha} C_{j\beta} \rangle = \frac{1}{m^2 \sqrt{NM}} \delta_{ij}\delta_{\alpha\beta} \tag{4.18}$$

Note that (2.3) and (4.16) are conventionally parametrized in such a way that (2.5) and (4.18) coincide.

The partition function for the corresponding Dyson gas reads[18]

$$Z = \prod_{\alpha=1}^{N} \int_0^\infty ds_\alpha \exp[-\tfrac{1}{2}\sqrt{NM}\, m^2 s_\alpha] \prod_{\beta=1}^{N} s_\beta^{(M-N-1)/2} \prod_{1 \leqslant \gamma < \delta \leqslant N} |s_\gamma - s_\delta| \tag{4.19}$$

[6] Note that the Riemann sheet of the square root in (4.13) is such that $\sqrt{(0-b^2)(0-a^2)} = -ab$, as we already observed in Section 2.

As before, the last two products constitute the Jacobian associated with polar coordinates. The energy functional $\mathscr{E}$ of the Dyson gas is now

$$\mathscr{E} = \frac{1}{2} \left\{ \sum_{\alpha=1}^{N} \left(m^2 s_\alpha - \frac{r-1-1/N}{\sqrt{r}} \log s_\alpha \right) - \frac{1}{N\sqrt{r}} \sum_{1 \leq \alpha < \beta \leq N} \log(s_\alpha - s_\beta)^2 \right\} \tag{4.20}$$

Thus, in the large-N, M limit, (4.20) becomes precisely one-half of the corresponding expression (4.5) for complex Hermitian matrices, and our discussion following (4.6) through (4.15) remains intact.

5. KAZAKOV'S METHOD EXTENDED TO RECTANGULAR COMPLEX MATRICES

5.1. Contour Integral

Gaussian matrix ensembles may be studied in many ways. Several years ago, Kazakov introduced a method[19] for treating the usual Gaussian ensemble of random Hermitian matrices which was later extended and applied to a study of random Hermitian matrices made of square blocks.[7] Here we generalize it to random Hermitian matrices made of rectangular blocks. It consists of adding to the probability distribution a matrix source, which will be set to zero at the end of the calculation, leaving us with a simple integral representation for finite N. As we will see, one cannot let the source go to zero before one reaches the final step. We modify the probability distribution (2.3) of the matrix[7] $C^\dagger C$ by adding a source A, an $N \times N$ Hermitian matrix with eigenvalues $(a_1,..., a_N)$:

$$P_A(C) = \frac{1}{Z_A} \exp(-\sqrt{MN} \operatorname{Tr} C^\dagger C - \sqrt{MN} \operatorname{Tr} A C^\dagger C) \tag{5.1}$$

Next we introduce the Fourier transform of the average resolvent with this modified distribution:

$$U_A(t) = \left\langle \frac{1}{N} \operatorname{Tr} e^{itC^\dagger C} \right\rangle_A \tag{5.2}$$

from which we recover the eigenvalue density

$$\tilde{\rho}(s) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-its} U_0(t) = \left\langle \frac{1}{N} \operatorname{Tr} \delta(s - C^\dagger C) \right\rangle \tag{5.3}$$

---

[7] For notational simplicity we set $m^2 = 1$ in (2.3) throughout this section.

of $C^\dagger C$ after setting the source A to zero. Without loss of generality we can assume that A is a diagonal matrix. Let us now calculate $U_A(t)$. We first integrate over the $N \times N$ unitary matrix U which diagonalizes $C^\dagger C$. This is done through the well-known Itzykson–Zuber integral over the unitary group[22]

$$\int dU \exp(\mathrm{Tr}\, A U B U^\dagger) = \frac{\det[e^{a_\alpha b_\beta}]}{\Delta(A)\,\Delta(B)} \tag{5.4}$$

where $\Delta(A)$ is the Vandermonde determinant constructed with the eigenvalues of A:

$$\Delta(A) = \prod_{\alpha < \beta} (a_\alpha - a_\beta) \tag{5.5}$$

$(b_1,\ldots, b_N)$ are the eigenvalues of B, and $\Delta(B)$ is the Vandermonde determinant built out of them. We are then led to

$$U_A(t) = \frac{1}{Z_A \Delta(A)} \frac{1}{N} \sum_{\alpha=1}^{N} \int ds_1 \cdots ds_N\, e^{i t s_\alpha} \Delta(s_1, \ldots, s_N)$$
$$\times \left(\prod_{\beta=1}^{N} s_\beta \right)^{M-N} \exp\left[-\sqrt{MN} \sum_{\gamma=1}^{N} s_\gamma (1 + a_\gamma) \right] \tag{5.6}$$

We now integrate over the s_α. It is easy to prove (for example, by using the Faddeev–Popov method) that

$$\int ds_1 \cdots ds_N\, \Delta(s_1,\ldots, s_N) \left(\prod_{\beta=1}^{N} s_\beta \right)^{M-N} \exp\left(-\sum_{\alpha=1}^{N} s_\alpha b_\alpha \right)$$
$$= C_N \frac{\Delta(b_1,\ldots, b_N)}{(\prod_1^N b_\alpha)^M} \tag{5.7}$$

where C_N is a constant independent of the b_α. Note that (5.7) is valid also for $M = N$. With the normalization $U_A(0) = 1$, we could always divide, at any intermediate step of the calculation, the expression we obtain for $U_A(t)$ by its value at $t = 0$, and thus the overall multiplicative factors in (5.6) and (5.7) are not needed.

We now apply this identity to the N terms of (5.6), with

$$b_\beta^{(\alpha)}(t) = \sqrt{MN} \left(1 + a_\beta - \frac{it}{\sqrt{MN}} \delta_{\alpha,\beta} \right) \tag{5.8}$$

and obtain

$$
\begin{aligned}
U_A(t) = \frac{1}{N} \sum_{\alpha=1}^{N} \prod_{\beta=1}^{N} & \left(\frac{1+a_\beta}{1+a_\beta - (it/\sqrt{MN})\,\delta_{\alpha,\beta}} \right)^M \\
& \times \prod_{\beta<\gamma} \frac{a_\beta - a_\gamma - (it/\sqrt{MN})(\delta_{\alpha,\beta} - \delta_{\alpha,\gamma})}{a_\beta - a_\gamma} \\
= \frac{1}{N} \sum_{\alpha=1}^{N} & \left(\frac{1+a_\alpha}{1+a_\alpha - it/\sqrt{MN}} \right)^M \prod_{\gamma \neq \alpha} \frac{a_\alpha - a_\gamma - it/\sqrt{MN}}{a_\alpha - a_\gamma}
\end{aligned} \tag{5.9}
$$

As a consistency check, note that for $M = N$, (5.9) coincides with Eq. (3.9) of ref. 7. This sum over N terms may be conveniently replaced by a contour integral in the complex plane:

$$
U_A(t) = \frac{i\sqrt{M/N}}{t} \oint \frac{du}{2\pi i} \left(\frac{1+u}{1+u - it/\sqrt{MN}} \right)^M \prod_{\gamma=1}^{N} \frac{u - a_\gamma - it/\sqrt{MN}}{u - a_\gamma} \tag{5.10}
$$

in which the contour encloses all the a_γ and no other singularity. It is now, and only now, possible to let all the a_γ go to zero. We thus obtain a simple expression for $U_0(t)$,

$$
U_0(t) = \frac{i\sqrt{M/N}}{t} \oint \frac{du}{2\pi i} \frac{(1 - it/u\sqrt{MN})^N}{(1 - it/(1+u)\sqrt{MN})^M} \tag{5.11}
$$

Note that this representation of $U_0(t)$ as a contour integral over one single complex variable is exact for any finite M, N, including $M = N = 1$.

5.2. The Density of States

In the large-M, N limit (with finite $r = M/N$), for finite t, the integrand in (5.11) becomes

$$
\exp\left[it \left(\frac{\sqrt{r}}{u+1} - \frac{1}{u\sqrt{r}} \right) \right]
$$

and therefore $U_0(t)$ approaches

$$
U_0(t) = \frac{\sqrt{r}}{it} \oint \frac{du}{2\pi i} \exp\left[it \left(\frac{\sqrt{r}}{1-u} + \frac{1}{u\sqrt{r}} \right) \right] \tag{5.12}
$$

where we changed u into $-u$.

Setting

$$z = \frac{1}{u\sqrt{r}} + \frac{\sqrt{r}}{1-u}$$

we change variables to

$$u = \frac{z - \sqrt{r} + 1/\sqrt{r} - \sqrt{(z - \sqrt{r} + 1/\sqrt{r})^2 - 4z/\sqrt{r}}}{2z} \qquad (5.13)$$

Then the integral of (5.12) becomes, after an integration by parts,

$$
\begin{aligned}
U_0(t) &= \frac{\sqrt{r}}{it} \oint \frac{dz}{2\pi i} \frac{du}{dz} e^{itz} \\
&= -\sqrt{r} \oint \frac{dz}{2\pi i} u(z) e^{itz} \\
&= -\sqrt{r} \oint \frac{dz}{2\pi i} \left[z - \sqrt{r} + \frac{1}{\sqrt{r}} - \sqrt{\left(z - \sqrt{r} + \frac{1}{\sqrt{r}}\right)^2 - \frac{4z}{\sqrt{r}}} \right] \frac{e^{itz}}{2z} \\
&= \frac{\sqrt{r}}{2\pi} \int_{b^2}^{a^2} \frac{dx}{x} \sqrt{(a^2 - x)(x - b^2)} \, e^{itx} \qquad (5.14)
\end{aligned}
$$

where a and b are given in (2.14). Therefore, we have from (5.3)

$$
\begin{aligned}
\tilde{p}(s) &= \int \frac{dt}{2\pi} e^{-itx} U_0(t) \\
&= \frac{\sqrt{r}}{2\pi} \frac{\sqrt{(a^2 - s)(s - b^2)}}{s} \qquad (5.15)
\end{aligned}
$$

for $b^2 \leqslant s \leqslant a^2$, and zero elsewhere. This expression coincides with (4.14) as expected. We observe from (1.9) that $p(\lambda)$ and $\tilde{p}(s) \equiv \tilde{p}(\lambda^2)$ are related by

$$p(\lambda) = \frac{r-1}{r+1} \delta(\lambda) + \frac{2|\lambda|}{r+1} \tilde{p}(\lambda^2) \qquad (5.16)$$

Substituting (5.15) into (5.16), we obtain (2.17) once again, as we should.

5.3. The Edges of the Eigenvalue Distribution

It is easy to apply this same method for studying the crossover at the edges of the eigenvalue distributions (2.17) or (5.15), namely in the vicinity of the endpoints $s = a^2$ and $s = b^2$. To this end, we observe from (5.3) and (5.11) that

$$\frac{\partial \tilde{\rho}(s)}{\partial s} = \sqrt{r} \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-its} \oint \frac{du}{2\pi i} \frac{(1 - it/Nu\sqrt{r})^N}{(1 - it\sqrt{r}/M(1+u))^M} \tag{5.17}$$

where the purpose of the s derivative is to get rid of the simple pole at $t = 0$ in (5.11). By changing t to $\sqrt{MN}\, t$ and then t to $t + iu$, as well as u to $-iu$, we obtain the factorized expression

$$\frac{\partial \tilde{\rho}(s)}{\partial s} = -iM \left(\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\sqrt{MN}\, ts} \frac{t^N}{(t+i)^M} \right)$$

$$\times \left(\oint \frac{du}{2\pi i} e^{i\sqrt{MN}\, us} \frac{(u+i)^M}{u^N} \right) \tag{5.18}$$

The advantage of (5.18) is that it is relatively easy to study its large-N, M behavior by saddle-point techniques. We observe that the t integral may be written as

$$I_{N, M} = \int_{-\infty}^{+\infty} dt\, e^{-\sqrt{NM}\, S_{\text{eff}}} \tag{5.19}$$

where S_{eff} is given by

$$S_{\text{eff}} = ist + \sqrt{r} \log(t+i) - \frac{1}{\sqrt{r}} \log t \tag{5.20}$$

Similarly, the integrand of the u integration is $\exp(\sqrt{NM}\, S_{\text{eff}})$. Thus, the large-$N$, M behavior of (5.17) is determined by the saddle points of a single function S_{eff}. Consider the t integral (5.19) first. It has two saddle points t_c at

$$t_c = \frac{s - ab \pm \sqrt{(s - a^2)(s - b^2)}}{2is} \tag{5.21}$$

where a, b are given in (2.14). The interesting situation occurs when these two saddle points become degenerate, namely at the endpoints $s = \lambda^2 = a^2$

and $s = \lambda^2 = b^2$. We thus investigate (5.17) at the vicinity of these points by focusing on these regions. Let us consider the neighborhood of $\lambda = a$ first (the crossover behavior around $\lambda = -a$ is simply the mirror image thereof.) We introduce the scaled variables

$$\lambda = a + N^{-\alpha}x$$
$$t = -i\frac{1}{\sqrt{r+1}} + N^{-\beta}\tau \tag{5.22}$$

with α, β to be determined, and expand S_{eff} up to τ^3. This leads to

$$S_{\text{eff}}(t) = S_* + 2r^{-1/4}N^{-\alpha}x + \tfrac{1}{3}ia^4\tau^3 N^{-3\beta} + 2iaN^{-\alpha-\beta}\tau x + \cdots \tag{5.23}$$

where S_* is the value of S_{eff} at the critical point, and the ellipsis stands for terms of $\mathcal{O}(N^{-2\alpha})$. We thus find that there is a large-N, finite-x limit, provided we fix the two unknown exponents α and β to

$$\alpha = 2/3, \qquad \beta = 1/3 \tag{5.24}$$

We repeat this for the u integral of (5.18). We then find that the leading terms of (5.23) of order 1, as well as the term $2xN^{-2/3}$, cancel with terms of opposite signs in the u integral. Thus we obtain the following equation for the density of state near the critical value $s = a^2$ or $\lambda = \pm a$:

$$\frac{\partial \bar{\rho}(\lambda^2)}{\partial \lambda^2} = -M^{1/3}\left(\frac{\sqrt{r}}{a^4}\right)^{2/3}\left|Ai\left[2\left(\frac{r}{a}\right)^{1/3}N^{2/3}(\lambda \mp a)\right]\right|^2 \tag{5.25}$$

where the Airy function $Ai(z)$ is defined as

$$Ai[(3\alpha)^{-1/3}x] = \frac{(3\alpha)^{1/3}}{\pi}\int_0^\infty \cos(\alpha t^3 + xt)\,dt \tag{5.26}$$

The Airy function in (5.25) is smoothly decreasing for $|\lambda| > a$, but it oscillates for $|\lambda| < a$.

Investigation of the behavior of (5.18) near the other critical points $\lambda = \pm b$ proceeds similarly. Concentrating on $\lambda = b$, we introduce the scaling variables

$$\lambda = b + N^{-\alpha}x$$
$$t = \frac{i}{\sqrt{r-1}} + N^{-\beta}\tau \tag{5.27}$$

and find that there is a large-N, finite-x limit, provided we fix the two unknown exponents α and β to the same values as before. Thus, the cross-over behavior of the density of states around $\lambda = \pm b$ is governed by the Airy function as well for any $r > 1$.

A new phenomenon appears, however, if we also take the limit $r \to 1$. It is easy to see, by rescaling τ in the expansion of S_{eff} into

$$T = (\sqrt{r} - 1)\,\tau \tag{5.28}$$

that the Airy function behavior of $\partial \bar{\rho}(\lambda^2)/\partial \lambda^2$ near $\lambda = \pm b$ breaks down as $r \to 1$. Indeed, from previous work[1-8] we know that the oscillations near the origin in the density of the eigenvalues of matrices built out of square blocks ($r = 1$) are governed by the Bessel function and not by the Airy function.

APPENDIX. THE CENTRAL LIMIT THEOREM— A RENORMALIZATION GROUP PROOF

As a simple, but perhaps amusing exercise we use the large-N renormalization group discussed in Section 3 to prove the celebrated central limit theorem of Gauss.

Consider a set of N independent random variables $\{x_1, x_2,..., x_N\}$ which are distributed according to some distribution function

$$Q_N(x_1,..., x_N) = \prod_{i=1}^{N} Q(x_i) \tag{A.1}$$

In order to be consistent with our normalization conventions in Section 3, we normalize this distribution function such that

$$\langle x_i \rangle = 0, \qquad \langle x_i x_j \rangle = \frac{\sigma^2}{N^{2\beta}} \delta_{ij} \tag{A.2}$$

where $\beta > 0$ is yet to be determined. Thus, a typical term drawn from $Q_N(x)$ is of the order $\sigma N^{-\beta}$. We wish to calculate the distribution function of the sum of these random numbers, namely, the quantity

$$P_N(s, \sigma) = \left\langle \delta \left(s - \sum_{i=1}^{N} x_i \right) \right\rangle_N \tag{A.3}$$

where $\langle \cdot \rangle_N$ denotes averaging with respect to $Q_N(x)$. In principle, P_N depends upon all the cumulants of $Q_N(x)$, but we expect that the large-N

limit of P_N will depend only upon σ. Following our discussion in Section 3, we now consider a set of $N + 1$ random variables whose distribution function $Q_{N+1}(x)$ is normalized such that

$$\langle x_i \rangle = 0, \qquad \langle x_i x_j \rangle = \frac{\sigma^2}{(N+1)^{2\beta}} \delta_{ij} \tag{A.4}$$

Then,

$$
\begin{aligned}
P_{N+1}(s, \sigma) &= \left\langle \delta\left(s - \sum_{i=1}^{N} x_i - x_{N+1}\right)\right\rangle_{N+1} \\
&= \left\langle \delta\left(s - \sum_{i=1}^{N} x_i\right)\right\rangle_{N+1} \\
&\quad + \frac{\sigma^2}{2(N+1)^{2\beta}} \frac{\partial^2}{\partial s^2} \left\langle \delta\left(s - \sum_{i=1}^{N} x_i\right)\right\rangle_{N+1} + \cdots
\end{aligned} \tag{A.5}
$$

where we used (A.4). The ellipsis stands for cumulants of order higher than two, which are clearly suppressed by powers of $N^{-\beta}$, and we neglect them henceforth. Comparing (A.2) and (A.4), we also see that

$$\left\langle \delta\left(s - \sum_{i=1}^{N} x_i\right)\right\rangle_{N+1} = P_N(s, \sigma') \tag{A.6}$$

with

$$\sigma' = \left(\frac{N}{N+1}\right)^\beta \sigma = \left(1 - \frac{\beta}{N}\right)\sigma + \cdots \tag{A.7}$$

We now use (A.6) and (A.7) to rewrite (A.5) as

$$P_{N+1}(s, \sigma) = \left(1 - \frac{\beta}{N}\sigma\frac{\partial}{\partial\sigma} + \frac{\sigma^2}{2N^{2\beta}}\frac{\partial^2}{\partial s^2}\right) P_N(s, \sigma) \tag{A.8}$$

where we neglected terms of $\mathcal{O}(1/N^{2\beta+1})$. We observe from (A.8) that variations of σ are as important as variations of s in the large-N limit only if

$$\beta = 1/2 \tag{A.9}$$

which fixes β. We thus conclude that

$$N \frac{\partial P_N}{\partial N} = \frac{\sigma}{2} \left(\sigma \frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial \sigma} \right) P_N(s, \sigma) \tag{A.10}$$

The left-hand side of (A.9) must vanish if P_N has a large-N limit

$$P_N(s, \sigma) \xrightarrow[N \to \infty]{} P(s, \sigma) \tag{A.11}$$

and thus

$$\left(\sigma \frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial \sigma} \right) P(s, \sigma) = 0. \tag{A.12}$$

A simple scaling argument similar to the one invoked in Section 3 leads to the relation

$$P(s, \sigma) = \frac{1}{\sigma} P\left(\frac{s}{\sigma}, 1 \right) \tag{A.13}$$

which implies that

$$\sigma \frac{\partial}{\partial \sigma} P = -P - s \frac{\partial}{\partial s} P \tag{A.14}$$

Substituting (A.14) in (A.12), we finally obtain the differential equation

$$\left(\sigma^2 \frac{\partial^2}{\partial s^2} + s \frac{\partial}{\partial s} + 1 \right) P(s, \sigma) = 0 \tag{A.15}$$

We solve (A.15) and find that its normalized solution is the Gaussian distribution

$$P(s, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{s^2}{2\sigma^2} \right) \tag{A.16}$$

which is the statement of the central limit theorem. The proof of the central limit theorem presented here is not any simpler than the conventional proof found in textbooks.

The generalization of this proof to the case[23] of adding a large number N of $K \times K$ matrices $\{\phi_1, ..., \phi_N\}$ is straightforward. In this case s and $P(s, \sigma)$ are $K \times K$ matrices. We take these matrices to be real (the Hermitian case can be treated similarly.) Then (A.15) becomes

$$\left(\sigma^2 \frac{\partial^2}{\partial s_\nu^\mu \, \partial s_\mu^\nu} + s_\nu^\mu \frac{\partial}{\partial s_\nu^\mu} + 1 \right) P(s, \sigma) = 0 \tag{A.17}$$

where μ, ν are indices of the $K \times K$ matrices (repeated indices are summed over.) The normalized solution of (A.17) is the Gaussian distribution

$$P(s, \sigma) = (\sqrt{2\pi} \, K\sigma)^{-K^2} \exp\left(-\frac{\mathrm{Tr}\, s^2}{2K^2\sigma^2} \right) \tag{A.18}$$

ACKNOWLEDGMENT

REFERENCES

1. J. J. M. Verbaarschot, *Nucl. Phys. B* **426**:559 (1994).
2. J. Ambjørn, Quantization of geometry, in *Les Houches 1990*, J. Dalibard *et al.*, eds. (North-Holland, Amsterdam, 1992), Section 4.3 and references therein.
3. J. Ambjørn, J. Jurkiewicz, and Yu. M. Makeenko, *Phys. Lett. B* **251**:517 (1990).
4. K. Slevin and T. Nagao, *Phys. Rev. Lett.* **70**:635 (1993); *Phys. Rev. B* **50**:2380 (1994); T. Nagao and K. Slevin, *J. Math. Phys.* **34**:2075, 2317 (1993).
5. T. Nagao and P. J. Forrester, *Nucl. Phys. B* **435**(FS):401 (1995).
6. A. V. Andreev, B. D. Simons, and N. Taniguchi, *Nucl. Phys. B* **432**:485 (1994).
7. E. Brézin, S. Hikami, and A. Zee, *Nucl. Phys. B* **464**:411 (1996).
8. S. Nishigaki, Preprint, hep-th/9606099.
9. J. J. M. Verbaarschot and I. Zahed, *Phys. Rev. Lett.* **70**:3852 (1993).
10. J. Jurkiewicz, M. A. Novak, and I. Zahed, preprint hep-ph/9603308; M. A. Novak, G. Papp and I. Zahed, preprint hep-ph/9603348.
11. S. Hikami and A. Zee, *Nucl. Phys. B* **446**:337 (1995).
12. S. Hikami, M. Shirai, and F. Wegner, *Nucl. Phys. B* **408**:415 (1993).
13. C. B. Hanna, D. P. Arovas, K. Mullen, and S. M. Girvin, cond-mat 9412102.
14. E. Brézin and A. Zee, *Phys. Rev. E* **49**:2588 (1994).
15. E. Brézin and A. Zee, *C. R. Acad. Sci. Paris* **317**:735 (1993).
16. E. Brézin and J. Zinn-Justin, *Phys. Lett. B* **288**:54 (1992); S. Higuchi, C. Itoh, S. Nishigaki, and N. Sakai, *Phys. Lett. B* **318**:63 (1993); *Nucl. Phys. B* **434**:283 (1995); Erratum, *Nucl. Phys. B* **441**:405.
17. J. D'Anna and A. Zee, *Phys. Rev. E* **53**:1399 (1996).
18. A. Anderson, R. C. Myers, and V. Periwal, *Phys. Lett. B* **254**:89 (1991); *Nucl. Phys. B* **360**:463 (1991); R. C. Myers and V. Periwal, *Nucl. Phys. B* **390**:716 (1991).

19. V. A. Kazakov, *Nucl. Phys. B* **354**:614 (1991).
20. E. P. Wigner, In *Canadian Mathematical Congress Proceedings* (University of Toronto Press, Toronto, 1957), p. 174 [reprinted in C. E. Porter, *Statistical Theories of Spectra: Fluctuation* (Academic Press, New York, 1965)]; see also M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
21. E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, *Commun. Math. Phys.* **59**:35 (1978).
22. C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21**:411 (1980).
23. A. Zee, *Nucl. Phys. B* **474**:726 (1996).